AFRL-AFOSR-VA-TR-2017-0018

Mixed-Integer Nonconvex Quadratic Optimization Relaxations and Performance Analysis

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01/21/2017
Final Report

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REPORT DOCUMENTATION PAGE

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1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From - To)*
14-09-2015	Final report	14-06-2012 to 14-06-2015

4. TITLE AND SUBTITLE
Mixed-Integer Nonconvex Quadratic Optimization Relaxations and Performance Analysis

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Zhi-Quan Luo

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
University of Minnesota
Sponsored Projects Administration
200 Oak Street, SE
Minneapolis, MN 55455

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Air Force Office of Scientific Research
875 North Randolph Street
Suite 325, Room 3112
Arlington VA, 22203

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

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13. SUPPLEMENTARY NOTES

14. ABSTRACT
The project addresses a fundamental question regarding the mixed integer quadratic programs (MIQP): how to find a provably high quality approximate solution efficiently? Given the nonconvex nature of the problem, two relaxation approaches are considered: one is based on convex semidefinite relaxation (SDR), while the other is based on quasi-convex relaxations. For SDR, a new probabilistic rounding procedure is proposed to account for both the binary and continuous variables. The performance of this rounding procedure is shown to deliver a constant factor approximation ratio for a class of the mixed integer quadratic optimization problems.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			**19b. TELEPHONE NUMBER** *(Include area code)*

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AFOSR Final Report
September 2015

Mixed-Integer Nonconvex Quadratic Optimization Relaxations and Performance Analysis

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Contents

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1 List of illustrations

2 Statement of the problem studied

This project considers a class of nonconvex quadratic optimization problems involving both integer and continuous variables. These nonconvex optimization problems are strongly motivated by applications in optimal motion planning, resource management of unmanned or micro aerial vehicles (UAV/MAVs) for joint target estimation and tracking, where collision avoidance and the mean squared estimation error minimization naturally lead to nonconvex quadratic constraints and a quadratic objective function. The integer variables arise due to assignment of UAV/MAVs to targets. This project also considers other applications and extensions of mixed integer quadratic optimization problem which include cardinality constrained quadratic programs (QP) and the low rank matrix completion problems.

The project addresses a fundamental question regarding the mixed integer quadratic programs (MIQP): how to find a provably high quality approximate solution efficiently? Given the nonconvex nature of the problem, two relaxation approaches are considered: one is based on convex semidefinite relaxation (SDR), while the other is based on quasi-convex relaxations. For SDR, a new probabilistic rounding procedure is proposed to account for both the binary and continuous variables. The performance of this rounding procedure will be analyzed in order to determine when the corresponding SDR can deliver a constant factor approximation ratio for the mixed integer nonconvex QPs. In contrast to the classical mixed integer nonlinear programming approaches, no convexity is assumed for the subproblem when some integer variables are fixed.

The following theoretic aspects of the mixed quadratic optimization problem have been studied.

- The focus of our study is on the approximation bounds of the SDP relaxation for both problems.

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The minimization model. Consider the following MBQCQP problem:

$$\begin{aligned}
\min_{\mathbf{w}\in\mathbb{F}^N,\boldsymbol{\beta}} \quad & \|\mathbf{w}\|^2 \\
\text{s.t.} \quad & \mathbf{w}^H\mathbf{H}_i\mathbf{w} \geq \beta_i \cdot 1 + (1-\beta_i)\cdot\epsilon,\ i\in\mathcal{M} \\
& \sum_{i\in\mathcal{M}} \beta_i \geq Q, \\
& \beta_i \in \{0,1\},\ i\in\mathcal{M}
\end{aligned} \tag{2.1}$$

where $\mathbb{F}$ is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, $\mathcal{M} = \{1,\cdots,M\}$, $\boldsymbol{\beta} = (\beta_1,\cdots,\beta_M)^T$, $\mathbf{H}_i$ $(i = 1,\cdots,M)$ are $N\times N$ real symmetric or complex Hermitian positive semidefinite matrices, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{F}^N$, M and Q are given integers satisfying $1\leq Q\leq M$, and ϵ is a given parameter satisfying $0\leq\epsilon\leq 1$. Throughout, we use the superscript H to denote the complex Hermitian transpose. Notice that the problem (2.1) can be easily solved either when $N = 1$ or $M = 1$, by solving a maximum eigenvalue problem. Hence, we shall assume that $N\geq 2$ and $M\geq 2$ in the rest of the paper. We note that problem (2.1) is in general NP-hard, due to the fact that one of its special cases with $Q = M$ is NP-hard.

The maximization model. Another interesting case of the MBQCQP problem takes the maximization form as follows:

$$\begin{aligned}
\max_{\mathbf{w}\in\mathbb{F}^n,\boldsymbol{\beta}} \quad & \|\mathbf{w}\|^2 \\
\text{s.t.} \quad & \mathbf{w}^H\mathbf{H}_i\mathbf{w} \leq \beta_i \cdot \epsilon + (1-\beta_i)\cdot 1,\ i\in\mathcal{M} \\
& \sum_{i\in\mathcal{M}} \beta_i \geq Q, \quad \beta_i \in \{0,1\},\ i\in\mathcal{M}
\end{aligned} \tag{2.2}$$

where $0\leq\epsilon\leq 1$ and $1\leq Q\leq M$. The above MBQCQP problem (2.2) arises naturally in the interference suppression problem in radar or wireless communication. Here, the interference suppression is captured by the constraints (2.2), in which the constants ϵ and 1 represent two distinctive suppression levels. The optimization problem becomes the one that maximizes the gain of the antenna array while suppressive undesirable interferences.

For these nonconvex mixed integer QPs, we propose Semidefinite relaxation approaches to solve these problems and analyse their approximation performance.

- In another related work, we study the problem of optimally partitioning the transmit nodes into cooperation groups of a wireless system, while at the same time designing their cooperation strategies. We focus on two related network settings in which either multiple nodes cooperatively transmit to a receiver, or a single node transmits to the receiver with the help of a set of cooperative relays. In both cases, the cooperative nodes are allowed to form a virtual antenna system, and they can jointly design the virtual transmit beamformers. More specifically, our objective is to find a subset of cooperative nodes (with given cardinality) and their joint linear beamformers so

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that the system performance measured by the receive signal to noise ratio (SNR) is maximized. We formulate the problem as a cardinality constrained quadratic program and study its computational complexity. Furthermore, we develop novel semi-definite relaxation (SDR) algorithms for this mixed integer quadratic program and prove that they have a guaranteed approximation performance in terms of the gap to global optimality, regardless of channel realization. Compared to the existing SDR algorithms and their analysis which focus on quadratic problems with continuous variables, our work deals with mixed-integer cardinality constrained quadratic optimization problems and therefore has a significantly broader scope.

These results provide not only useful insights on the semidefinite relaxation strategy for the mixed integer quadratic optimization but also simple resource allocation and user-base station association algorithms that are practically implementable in a large scale military MIMO communication system.

3 Summary of the most important results

Significant progress has been made on several fronts:

3.1 Semidefinite approximation for mixed binary quadratically constrained quadratic programs

Motivated by applications in wireless communications, this work develops semidefinite programming (SDP) relaxation techniques for some mixed binary quadratically constrained quadratic programs (MBQCQP) and analyzes their approximation performance. We consider both a minimization and a maximization model of this problem. For the minimization model, the objective is to find a minimum norm vector in N-dimensional real or complex Euclidean space, such that M concave quadratic constraints and a cardinality constraint are satisfied with both binary and continuous variables. By employing a special randomized rounding procedure, we show that the ratio between the norm of the optimal solution of the minimization model and its SDP relaxation is upper bounded by $\mathcal{O}(Q^2(M-Q+1)+M^2)$ (resp. $\mathcal{O}(Q^2(M-Q+1))$) in the real case and by $\mathcal{O}(M(M-Q+1))$ (resp. $\mathcal{O}(Q(M-Q+1))$) in the complex case when the given parameter ϵ satisfies $0<\epsilon<1$ (resp. when $\epsilon=0$). For the maximization model, the goal is to find a maximum norm vector subject to a set of quadratic constraints and a cardinality constraint with both binary and continuous variables. We show that in this case the approximation ratio is bounded from below by $\mathcal{O}(\epsilon/\ln(M))$ for both the real and the complex cases. Moreover, this ratio is tight up to a constant factor in general case.

Table 1 shows the average ratio (mean) of $v^{\min}_{\text{UBQP}}/v^{\min}_{\text{SDP}}$ over 300 independent realizations of i.i.d. real-valued Gaussian $\mathbf{h}_i$, $(i=1,\cdots,M)$ for several combinations of M, Q and N. The maximum value (max) and the standard deviation (Std) of $v^{\min}_{\text{UBQP}}/v^{\min}_{\text{SDP}}$ over 300 independent realizations are also shown in Table 1. Table 2 shows the corresponding average value, maximum value and the standard deviation of $v^{\min}_{\text{UBQP}}/v^{\min}_{\text{SDP}}$ for $\mathbb{F}=\mathbb{C}$. These results are significantly better than what is predicted by the worst-case analysis. In all test examples, the average values of $v^{\min}_{\text{UBQP}}/v^{\min}_{\text{SDP}}$ are lower than 4 (resp. lower than 3) when $\mathbb{F}=\mathbb{R}$ (resp. when $\mathbb{F}=\mathbb{C}$).

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Table 1. Mean and standard deviation of the approximation ratio over 300 independent realizations of real Gaussian i.i.d. $\mathbf{h}_i$ $(i = 1, \cdots, M)$, when $\mathbb{F} = \mathbb{R}$.

M	Q	$N = 4$			$N = 8$		
		max	mean	Std	max	mean	Std
$M = 8$	$Q = M/4$	3.7394	2.0348	0.2266	4.3387	2.0392	0.2948
	$Q = M/2$	3.9420	1.7972	0.1828	3.5232	1.7378	0.1475
	$Q = 3M/4$	4.6973	1.7863	0.3921	4.5721	1.8130	0.3428
$M = 12$	$Q = M/4$	4.9450	2.2191	0.2451	3.9625	2.1710	0.2304
	$Q = M/2$	5.8068	2.0639	0.4564	4.3483	2.0204	0.3241
	$Q = 3M/4$	7.7829	2.5970	1.3075	9.7150	2.8277	1.9578
$M = 16$	$Q = M/4$	4.2703	2.2977	0.2410	4.2980	2.2117	0.1972
	$Q = M/2$	7.3115	2.4463	0.9348	7.8240	2.4166	1.1345
	$Q = 3M/4$	10.715	3.2272	2.3823	10.621	3.7786	2.8760

Table 2. Mean and standard deviation of upper bound ratio over 300 independent realizations of real Gaussian i.i.d. $\mathbf{h}_i$ $(i = 1, \cdots, M)$, when $\mathbb{F} = \mathbb{C}$.

M	Q	$N = 4$			$N = 8$		
		max	mean	Std	max	mean	Std
$M = 8$	$Q = M/4$	4.8049	2.3720	0.2790	4.3579	2.4239	0.2757
	$Q = M/2$	3.7344	1.9308	0.1443	3.4477	1.9243	0.1477
	$Q = 3M/4$	2.7549	1.5812	0.0769	2.4477	1.5860	0.0818
$M = 12$	$Q = M/4$	4.0557	2.4986	0.1938	3.7851	2.4657	0.1998
	$Q = M/2$	3.2911	2.0301	0.1483	3.2867	2.0567	0.1190
	$Q = 3M/4$	2.6007	1.6451	0.0800	3.1609	1.6693	0.0860
$M = 16$	$Q = M/4$	3.8170	2.5778	0.1647	4.2616	2.5852	0.1757
	$Q = M/2$	3.6268	2.0908	0.0932	3.7761	2.0729	0.1065
	$Q = 3M/4$	2.9218	1.8024	0.1044	3.6056	1.8344	0.1432

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3.2 Joint User Grouping and Linear Virtual Beamforming: Complexity, Algorithms and Approximation Bounds

In a wireless system with a large number of distributed nodes, the quality of communication can be greatly improved by pooling the nodes to perform joint transmission/reception. In this work, we consider the problem of optimally selecting a subset of nodes from potentially a large number of candidates to form a virtual multi-antenna system, while at the same time designing their joint linear transmission strategies. We focus on two specific application scenarios: *1)* multiple single antenna transmitters cooperatively transmit to a receiver; *2)* a single transmitter transmits to a receiver with the help of a number of cooperative relays. We formulate the joint node selection and beamforming problems as *cardinality constrained optimization problems* with both discrete variables (used for selecting cooperative nodes) and continuous variables (used for designing beamformers). For each application scenario, we first characterize the computational complexity of the joint optimization problem, and then propose novel semi-definite relaxation (SDR) techniques to obtain approximate solutions. We show that the new SDR algorithms have a guaranteed approximation performance in terms of the gap to global optimality, regardless of channel realizations. The effectiveness of the proposed algorithms is demonstrated via numerical experiments.

In Fig. 1–2 we plot the performance of the proposed relaxation algorithms for different sizes of the network. For a given network size, we choose $Q = 10$ and let $N = 5$. For each network (Q, M) pair, the algorithm is run for 500 independent realizations of the network. We again plot the maximum, the minimum and the averaged approximation ratios achieved among those 500 realizations. We see that the proposed algorithm achieves very low worst-case approximation ratio, which suggests that high SNR performance is obtained for almost all Monte Carlo runs.

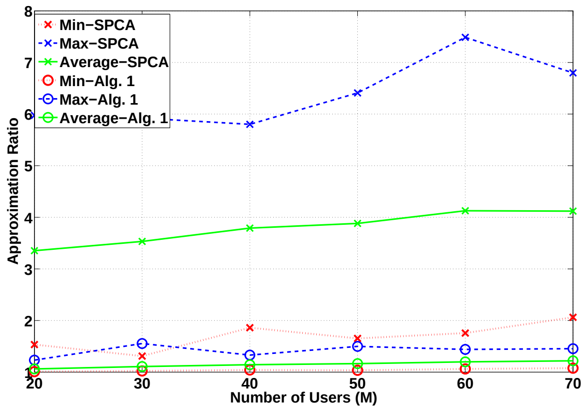

Figure 1. Approximation ratio for admission control with different network sizes. $M \in [10, 20, 30, 40, 50, 60, 70]$, $Q = 10$, $P = -10$dBW, $N = 5$.

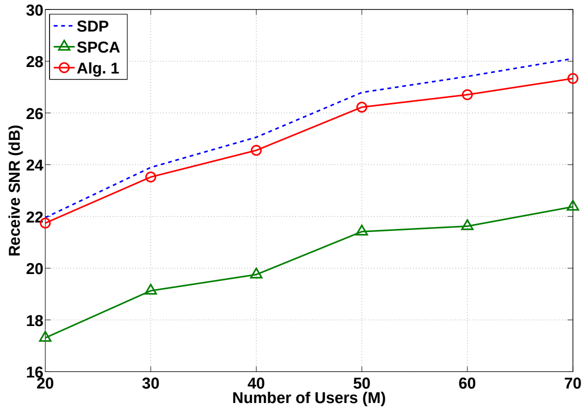

Figure 2. Receive SNR for admission control with different network sizes. $M \in [10, 20, 30, 40, 50, 60, 70]$, $Q = 10$, $P = -10$dBW, $N = 5$.

4 List of Publications Supported by this Project

(I) **Papers published or accepted for publication in peer-reviewed journals**

(1) "Joint User Grouping and Linear Virtual Beamforming: Complexity, Algorithms and Approximation Bounds"

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- *Authors:* Hong, M., Xu, Z., Razaviyayn, M. and Luo, Z.-Q.
- *IEEE Journal on Selected Areas in Communications Special Issue on Virtual MIMO*, Vol. 30, pp. 2013–2027, 2013.

(2) "Semidefinite Approximation for Mixed Binary Quadratically Constrained Quadratic Programs"
- *Authors:* Xu, Z., Hong, M. and Luo, Z.-Q.
- Accepted for publication in *SIAM Journal on Optimization*, Vol. 24, pp. 1265-1293, 2014.

(II) **Papers submitted, but not published**

(1) "On the linear convergence of alternating direction method of multipliers"
- *Authors:* Mingyi Hong and Zhi-Quan Luo
- Submitted to *Mathematical Programming,* Series A.

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Mixed-Integer Nonconvex Quadratic Optimization: Relaxations and Performance Analysis

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The full name of the principal investigator on the grant or contract.

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Abstract

The project addresses a fundamental question regarding the mixed integer quadratic programs (MIQP): how to find a provably high quality approximate solution efficiently? Given the nonconvex nature of the problem, two relaxation approaches are considered: one is based on convex semidefinite relaxation (SDR), while the other is based on quasi-convex relaxations. For SDR, a new probabilistic rounding procedure is proposed to account for both the binary and continuous variables. The performance of this rounding procedure will be analyzed in order to determine when the corresponding SDR can deliver a constant factor approximation ratio for the mixed integer nonconvex QPs. In contrast to the classical mixed integer nonlinear programming approaches, no convexity is assumed for the subproblem when some integer variables are fixed.

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Program Officer

Research Objectives

Technical Summary

Funding Summary by Cost Category (by FY, $K)

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